7.1 Integration By Parts

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1 Integration By Parts

To differentiate a product of two functions of x, one uses the Product Rule for derivatives,

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x).$$

$$\text{Thus, } \int \left[f(x)g'(x) + g(x)f'(x)\right]\, dx = f(x)g(x)$$

$$\int f(x)g'(x)dx + \int g(x)f'(x)\, dx = f(x)g(x)$$

$$\text{So, } \int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx$$

Note 1: The formula $\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx$ is called ***formula for integration by parts.*** An easier formula to remember is the following: Let

$$u = f(x) \qquad dv = g'(x)dx$$

$$du = f'(x)\, dx \qquad v = g(x)$$

$$\int udv = uv - \int vdu$$

Guidelines for Integration by Parts $\int u\, dv$

How to choose u and dv.

1. Let u be a factor of the integrand that becomes simpler (less complicated-reduced) when differentiated.
2. Let dv be a factor of the integrand that can be integrated easily(fits a basic integration rule).
3. The selection of u and dv must make the second integral $\int v\, du$ less complicated.

When to use Integration by Parts

1. The product of polynomial or power with trigonometric function or inverse trigonometric function where the polynomial or the power is not the derivative of the angle. For example.

Use Integration By Parts	Do not Use It
$\int x^n \sin(bx)\,dx$	$\int x\sin(x^2)\,dx$
$\int \sqrt[n]{x}\cos(bx)\,dx$	$\int x^2\cos(x^3)\,dx$
$\int x^n \sec^2(bx)\,dx$	$\int x\sec^2(x^2)\,dx$
$\int \sin^{-1}x\,dx$	
$\int \tan^{-1}x\,dx$	
$\int \sin^n x\,dx$	
$\int \cos^m x\,dx$	

2. The product of polynomial or power with exponential function where the polynomial or the power is not the derivative of the exponent. For example.

Use Integration By Parts	Do not Use It
$\int x^n e^{bx}\,dx$	$\int x3^{x^2}\,dx$
$\int \sqrt[n]{x}5^{bx}\,dx$	$\int x^2 6^{x^3}\,dx$
$\int (x^n+a)e^{bx}\,dx$	$\int xe^{x^2}\,dx$
$\int \sqrt[n]{x}e^{x}\,dx$	

3. The product of polynomial or power with logarithmic function where the polynomial or the power is not the derivative of the function inside the log or ln . For example.

Use Integration By Parts	Do not Use It
$\int \ln x\,dx$	$\int \frac{\ln x}{x}\,dx$
$\int (\ln x)^m\,dx$	$\int \frac{\log_2 x}{x}\,dx$
$\int (x^n+a)\ln x\,dx$	
$\int \sqrt[n]{x}\ln x\,dx$	

4. The product of exponential function with trigonometric function where the exponential function is not the derivative of the angle. For example.

Use Integration By Parts	Do not Use It
$\int e^x\cos x\,dx$	$\int e^x\cos(e^x)\,dx$
$\int e^{ax}\sin(bx)\,dx$	$\int e^{2x}\sin(e^{2x})\,dx$

Example 1: Find $\displaystyle\int x\cos x\,dx$.

Solution:

Let

$$u = x \qquad dv = \cos x\,dx$$

$$du = dx \qquad v = \sin x$$

Therefore,

$$\begin{aligned}\int x\cos x\,dx &= \underbrace{(x)(\sin x)}_{uv} - \underbrace{\int \sin x\,dx}_{\int v\,du}\\ &= x\sin x - \int \sin x\,dx\\ &= x\sin x - (-\cos x) + C\\ &= x\sin x + \cos x + C.\end{aligned}$$

Example 2: Find $\displaystyle\int \ln x\,dx$.

Solution:

Let

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad v = x$$

Therefore,

$$\begin{aligned}\int \ln x\,dx &= \underbrace{(\ln x)(x)}_{uv} - \underbrace{\int x\frac{1}{x}\,dx}_{\int v\,du}\\ &= x\ln x - \int dx\\ &= x\ln x - x + C.\end{aligned}$$

Example 3: Find $\displaystyle\int t^2 e^t\,dt$.

Solution:

Let

$$u = t^2 \qquad dv = e^t\,dt$$

$$du = 2t\,dt \qquad v = e^t$$

Therefore,

$$\int t^2e^t\,dt = t^2e^t - \int 2te^t\,dt \qquad \text{Use } \int u\,dv = uv - \int v\,du\ .$$

$$= t^2e^t - 2\int te^t\,dt \qquad \text{Int. by part again } u = t \quad dv = e^t\,dt.$$

$$du = dt \quad v = e^t.$$

$$= t^2e^t - 2\left[te^t - \int e^t\,dt\right] \qquad \text{integrate } \int e^t\,dt = e^t\ .$$

$$= t^2e^t - 2\left[te^t - e^t\right] + C \qquad \text{simplify}$$

$$t^2e^t - 2te^2 + 2e^t + C$$

$$= (t^2 - 2t + 2)e^t + C.$$

Example 4: Find $\int e^x \sin x\,dx$.

Solution:

Let

$$u = \sin x \qquad dv = e^x\,dx$$

$$du = \cos x\,dx \qquad v = e^x$$

Therefore,

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx \qquad \text{Use } \int u\,dv = uv - \int v\,du\ .$$

$$= e^x \sin x - \int e^x \cos x\,dx \qquad \text{Int. by part again } u = \cos x \quad dv = e^x\,dx.$$

$$du = -\sin x\,dx \quad v = e^x.$$

$$= e^x \sin x - \left[e^x \cos x - \int (e^x)(-\sin x)\,dx\right] \qquad \text{Use } \int u\,dv = uv - \int v\,du\ .$$

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx \qquad \text{We are trying to find } \int e^x \sin x\,dx.$$

$$\int e^x \sin x\,dx + \int e^x \sin x\,dx = e^x \sin x - e^x \cos x \qquad \text{Add the two integral.}$$

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x \qquad \text{Divide by 2 and add } C \text{ to the right side.}$$

$$\int e^x \sin x\,dx = \frac{e^x \sin x - e^x \cos x}{2} + C.$$

Example 5: Find $\int_0^1 \tan^{-1} x\,dx$.

Solution:

Let

$$u = \arctan x \qquad dv = dx$$

$$du = \frac{1}{1+x^2}\,dx \qquad v = x$$

Therefore,

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx \qquad \text{Note } d(1+x^2) = 2x\,dx\ .$$

$$= \left[(1)\tan^{-1} 1 - (0)\tan^{-1} 0\right] - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx \qquad \text{Use } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|\ .$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$

$$= \frac{\pi - \ln 4}{4}.$$

Reduction Formulas

Integration by parts can be used to derive reduction formulas for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a lower power of that function.

1. $\displaystyle\int \sin^n x\,dx = \frac{-1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$
2. $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$
3. $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx,\ n \neq 1,\ \int \sec x\,dx = \ln|\sec x + \tan x|$
4. $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx,\ n \neq 1,\ \int \tan x\,dx = \ln|\sec x|$
5. $\displaystyle\int \cot^n x\,dx = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx,\quad n \neq 1,\ \int \cot x\,dx = \ln|\sin x|$
6. $\displaystyle\int \csc^n x\,dx = \frac{\csc^{n-2} x\cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx,\ n \neq 1,\ \int \csc x\,dx = \ln|\csc x - \cot x|$

Example 6: Prove the reduction formula $\displaystyle\int \sin^n x\,dx = \frac{-1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

Solution:

Let

$$u = (\sin x)^{n-1} \qquad dv = \sin x\, dx$$

$$du = (n-1)(\sin x)^{n-2} \cos x\, dx \qquad v = -\cos x$$

Thus, we use $\cos^2 x = 1 - \sin^2 x.$,

$$\begin{aligned}\int \sin^n x\, dx &= \int \sin^{n-1} x \sin x\, dx \\ &= \sin^{n-1} x(-\cos x) - \int (-\cos x)(n-1)(\sin^{n-2} x) \cos x\, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x(1 - \sin^2 x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx\end{aligned}$$

$$\overbrace{(n-1) \int \sin^n x\, dx + \int \sin^n x\, dx}^{n \int \sin^n x\, dx} = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx$$

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx.$$

Example 7: Evaluate $\int \sin(\sqrt{x})\, dx$

Solution:

Let $w = \sqrt{x} \Rightarrow w^2 = x,$ so $2wdw = dx.$

Hence $\int \sin(\sqrt{x})\, dx = \int \sin w\, 2wdw = 2 \int w \sin w\, dw.$

Now, using integration by parts with

$$u = w \qquad dv = \sin w\, dw$$

$$du = dw \qquad v = -\cos w$$

Therefore,

$\int w \sin w\, dw = -w \cos w - \int (-\cos w)\, dw = -w \cos w + \int \cos w\, dw = -w \cos w + \sin w + C.$

$\int \sin(\sqrt{x})\, dx = 2 \int w \sin w\, dw = -2w \cos w + 2 \sin w + C.$

Thus, $\int \sin(\sqrt{x})\, du = -2\sqrt{x} \cos(\sqrt{x}) + 2 \sin(\sqrt{x}) + C$

Example 8: Find $\int \frac{x^3}{\sqrt{1+x^2}}\, dx.$

Solution:

Using change of variable,

let $u = \sqrt{1+x^2}$, then $u^2 = 1 + x^2$,

$2udu = 2xdx$.

Thus $xdx = udu$ and $x^2 = u^2 - 1$.

Hence

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1+x^2}} xdx &= \int \frac{u^2-1}{\cancel{u}} \cancel{u} du \\
&= \int (u^2-1)\, du \\
&= \frac{1}{3}u^3 - u + C \\
&= \frac{1}{3}\sqrt{(1+x^2)^3} - \sqrt{1+x^2} + C \\
&= \frac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C \\
&= (1+x^2)^{1/2}\left[\frac{1}{3}(1+x^2) - 1\right] + C \\
&= (1+x^2)^{1/2}\left[\frac{1}{3}x^2 - \frac{2}{3}\right] + C \\
&= \frac{(x^2-2)\sqrt{x^2+1}}{3} + C
\end{aligned}$$

Using Integration by parts,

$$u = x^2 \quad dv = (1+x^2)^{-1/2}\, xdx$$

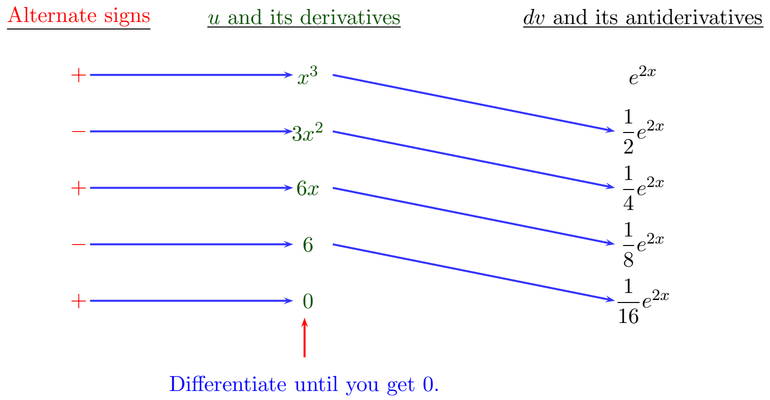
$$du = 2x\, dx \quad v = \sqrt{1+x^2}$$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1+x^2}} dx &= x^2\sqrt{1+x^2} - \int 2x\sqrt{1+x^2}\, dx \\
&= x^2\sqrt{1+x^2} - \int (1+x^2)^{1/2} 2xdx \\
&= x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2} + C \\
&= x^2\sqrt{1+x^2} - \frac{2}{3}\sqrt{(1+x^2)^3} + C \\
&= x^2(1+x^2)^{1/2} - \frac{2}{3}(1+x^2)^{2/3} + C \\
&= (x^2+1)^{1/2}\left[x^2 - \frac{2}{3}(1+x^2)\right] + C \\
&= (1+x^2)^{1/2}\left[\frac{1}{3}x^2 - \frac{2}{3}\right] + C \\
&= \frac{(x^2-2)\sqrt{x^2+1}}{3} + C
\end{aligned}$$

Integrals of the form $\int f(x)\,g(x)\,dx$, in which the higher derivative of f becomes zero and the antiderivatives of g can be calculated without difficulty, can be evaluated using tabular integration.

Example 9: Evaluate $\int x^3 e^{2x}\,dx$

Solution:



$$\int x^3 e^{2x} dx = +\frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{6}{8}xe^{2x} - \frac{6}{16}e^{2x} + C = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C.$$

7.2 Trigonometric Integrals

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2 Trigonometric Integrals

Integrating Powers of the Sine and Cosine Functions

Trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$	2. $\sin 2x = 2\sin x \cos x$
3. $\sin^2 x = \dfrac{1-\cos 2x}{2}$	4. $\cos^2 x = \dfrac{1+\cos 2x}{2}$
5. $\sin x \cos y = \dfrac{1}{2}[\sin(x-y)+\sin(x+y)]$	6. $\sin x\, \sin y = \dfrac{1}{2}[\cos(x-y)-\cos(x+y)]$
7. $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$	8. $\cos x \cos y = \dfrac{1}{2}[\cos(x-y)+\cos(x+y)]$.

Guidelines for Evaluating $\int \sin^m x \cos^n x\, dx$

1. If the power of the $\sin x$, $m = 2k+1$ is odd and positive, save one $\sin x$ factor and convert the remaining factors to $\cos x$. Then, expand and integrate. We assume that $u = \cos x$, then $du = -\sin x\, dx$.

$$\begin{aligned}\int \overbrace{\sin^{2k+1}}^{\text{odd}} x \cos^n x\, dx &= \int \overbrace{(\sin^2 x)^k}^{\text{convert to }\cos x} \cos^n x \overbrace{\sin x\, dx}^{\text{save for } du} \\ &= \int (1-\cos^2 x)^k \cos^n x \sin x\, dx \\ &= -\int (1-u^2)^k u^n\, du\end{aligned}$$

2. If the power of the $\cos x$, $n = 2k+1$ is odd and positive, save one $\cos x$ factor and convert the remaining factors to $\sin x$. Then, expand and integrate. We assume that $u = \sin x$, then

$du = \cos x\, dx$.

$$\int \sin^m x \overbrace{\cos^{2k+1} x}^{\text{odd}}\, dx = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{convert to } \sin x} \overbrace{\cos x\, dx}^{\text{save for } du}$$
$$= \int \sin^m x(1-\sin^2 x)^k \cos x\, dx$$
$$= \int u^m (1-u^2)^k\, du$$

3. If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities (3) and (4).

Example 1: Find $\int \cos^3 x\, dx$.

Solution:

$$\int \cos^3 x\, dx = \int \cos^2 x\ \cos x\, dx$$
$$= \int (1-\sin^2 x)\ \cos x dx$$
$$= \int (1-\sin^2 x)\ \cos x\, dx \quad \text{Let } u = \sin x, \text{ then } du = \cos x\, dx.$$
$$= \int (1-u^2)\, du$$
$$= u - \frac{1}{3}u^3 + C$$
$$= \sin x - \frac{1}{3}\sin^3 x + C.$$

Example 2: Find $\int \sin^5 x \cos^2 x\, dx$.

Solution:

$$\begin{aligned}
\int \sin^5 x \cos^2 x \, dx &= \int \sin^4 x \cos^2 x \, \sin x \, dx \\
&= \int (\sin^2 x)^2 \cos^2 x \, \sin x \, dx \\
&= \int (1-\cos^2 x)^2 \cos^2 x \, \sin x dx \qquad \text{Let } u=\cos x, \text{ then } du=-\sin x \, dx. \\
&= \int (1-u^2)^2 u^2 \, (-du) \\
&= -\int (1-2u^2+u^4) u^2 \, du \\
&= -\int (u^2-2u^4+u^6) \, du \\
&= -\left[\frac{u^3}{3}-\frac{2u^5}{5}+\frac{u^7}{7}\right]+C \\
&= \frac{-1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x + C.
\end{aligned}$$

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Example 3: Find $\int_0^{\pi} \sin^2 x \, dx$.

Solution:

$$\begin{aligned}
\int_0^{\pi} \sin^2 x \, dx &= \int_0^{\pi} \sin^2 x \, dx \qquad \sin^2 x = \frac{1-\cos(2x)}{2}. \\
&= \int_0^{\pi} \frac{1-\cos(2x)}{2} \, dx \\
&= \frac{1}{2}\int_0^{\pi} [1-\cos(2x)] \, dx \qquad \int \cos(ax) \, dx = \frac{1}{a}\sin(ax)+C \\
&= \frac{1}{2}\left[x-\frac{1}{2}\sin(2x)\right]_0^{\pi} \\
&= \frac{1}{2}\left[(\pi-\frac{1}{2}\sin(2\pi))-(0-\frac{1}{2}\sin(2(0)))\right] \\
&= \frac{\pi}{2}
\end{aligned}$$

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Example 4: Find $\int \sin^4 x \, dx$.

Solution:

$$\begin{aligned}
\int \sin^4 x\,dx &= \int (\sin^2 x)^2\,dx && \sin^2 x = \frac{1-\cos(2x)}{2}. \\
&= \int \left[\frac{1-\cos(2x)}{2}\right]^2 dx \\
&= \frac{1}{4}\int [1-\cos(2x)]^2\,dx \\
&= \frac{1}{4}\int [1-2\cos(2x)+\cos^2(2x)]\,dx && \cos^2(2x) = \frac{1+\cos(4x)}{2}. \\
&= \frac{1}{4}\int [1-2\cos(2x)+\frac{1}{2}+\frac{1}{2}\cos(4x)]\,dx \\
&= \frac{1}{4}\int [\frac{3}{2}-2\cos(2x)+\frac{1}{2}\cos(4x)]\,dx && \int \cos(ax)\,dx = \frac{1}{a}\sin(ax)+C \\
&= \frac{1}{4}\left(\frac{3}{2}x-2\frac{1}{2}\sin(2x)+\frac{1}{2}\frac{1}{4}\sin(4x)\right)+C \\
&= \frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C
\end{aligned}$$

Integrating Powers and Products of the Tangent and Secant Functions

Useful trigonometric identity: $\tan^2 x + 1 = \sec^2 x$

Useful integrals:

1. $\int \sec x \tan x\,dx = \sec x + C$	2. $\int \sec^2 x\,dx = \tan x + C$
3. $\int \tan x\,dx = \ln\lvert\sec x\rvert + C$	4. $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C.$

To see $\int \sec x\,dx = \ln|\sec x + \tan x| + C.$

$$\begin{aligned}
\int \sec x\,dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx && (\sec x + \tan x)' = \sec x \tan x + \sec^2 x. \\
&= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx && \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \\
&= \ln|\sec x + \tan x| + C
\end{aligned}$$

Guidelines for Evaluating $\int \tan^m x \sec^n x\, dx$

1. If the power of the $\tan x$, $m = 2k+1$ is odd and positive, save a $\sec x \tan x$ factor and convert the remaining factors to $\sec x$. Then, expand and integrate. We assume that $u = \sec x$, then $du = \sec x \tan x\, dx$.

$$\int \overbrace{\tan^{2k+1}}^{\text{odd}} x \sec^n x\, dx = \int \overbrace{(\tan^2 x)^k}^{\text{convert to } \sec x} \sec^{n-1} \overbrace{x \sec x \tan x\, dx}^{\text{save for } du}$$
$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\, dx = \int (u^2-1)^k u^{n-1}\, du.$$

2. If the power of the $\sec x$, $m = 2k$ is even and positive, save a $\sec^2 x$ factor and convert the remaining factors to $\tan x$. Then, expand and integrate. We assume that $u = \tan x$, then $du = \sec^2 x\, dx$.

$$\int \tan^m x \overbrace{\sec^{2k}}^{\text{even}} x\, dx = \int \tan^m x \overbrace{(\sec^2 x)^{k-1}}^{\text{convert to } \tan x} \overbrace{\sec^2 x\, dx}^{\text{save for } du}$$
$$= \int \tan^m x\, (1+\tan^2 x)^{k-1} \sec^2 x\, dx$$
$$= \int u^m\, (1+u^2)^{k-1}\, du.$$

3. If the power of the $\tan x$ is an even positive and the power of the $\sec x$ is an odd positive convert to $\sec x$ and use the reduction formula or possibly use integration by parts

Example 5: Find $\int \tan^6 x \sec^4 x\, dx$.

Solution:

$$\int \tan^6 x \sec^4 x\, dx = \int \tan^6 x \sec^2 x\ \sec^2 x\, dx$$
$$= \int \tan^6 x (1+\tan^2 x)\ \sec^2 x dx$$
$$= \int u^6 (1+u^2)\, du \qquad u = \tan x \quad du = \sec^2 x\, dx.$$
$$= \int (u^6+u^8)\, du$$
$$= \frac{1}{7}u^7 + \frac{1}{9}u^9 + C$$
$$= \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C.$$

Example 6: Find $\int \tan^5 \theta \sec^7 \theta\, d\theta$.

Solution:

$$\begin{aligned}\int \tan^5\theta \sec^7\theta\, d\theta &= \int \tan^4\theta \sec^6\theta\, \sec\theta\tan\theta\, d\theta \\ &= \int (\tan^2\theta)^2 \sec^6\theta\, \sec\theta\tan\theta\, d\theta \\ &= \int (\sec^2\theta - 1)^2 \sec^6\theta\, \sec\theta\tan\theta\, d\theta \\ &= \int (u^2-1)^2 u^6\, du \qquad u=\sec\theta \quad du=\sec\theta\tan\theta\, d\theta. \\ &= \int (u^4-2u^2+1)u^6\, du \\ &= \int (u^{10}-2u^8+u^6)\, du \\ &= \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C \\ &= \frac{1}{11}\sec^{11}\theta - \frac{2}{9}\sec^9\theta + \frac{1}{7}\sec^7\theta + C.\end{aligned}$$

Example 7: Find $\int \tan^3 x\, dx$.

Solution:

$$\begin{aligned}\int \tan^3 x\, dx &= \int \tan^2 x \tan x\, dx \qquad \text{Use } \tan^2 x = \sec^2 x - 1\ . \\ &= \int (\sec^2 x - 1)\tan x\, dx \\ &= \int \overbrace{\tan x}^{f(x)}\overbrace{\sec^2 x}^{f'(x)} dx - \int \tan x\, dx \qquad \text{Use } \int f(x)f'(x)dx = \frac{[f(x)]^2}{2} + C\ . \\ &= \frac{\tan^2 x}{2} - \ln|\sec x| + C \\ \int \tan^3 x\, dx &= \frac{\tan^2 x}{2} - \ln|\sec x| + C.\end{aligned}$$

Example 8: Find $\int \sec^3 x\, dx$.

Solution:

Notice that $\int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx$ Let

$u = \sec x$ $\qquad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx$ $\qquad v = \tan x$

Therefore,

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx \qquad \text{Use } \int u\,dv = uv - \int v\,du\,.$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx \qquad 1 + \tan^2 x = \sec^2 x.$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \qquad \text{Collect the like integrals.}$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| \qquad \text{Divide by 2 and add } C \text{ to the right side.}$$

$$\int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Integrals Involving Cotangent and Cosecant

The guidelines for integrals involving cotangent and cosecant would be similar to that of integrals involving tangent and secant.

Useful trigonometric identity: $\cot^2 x + 1 = \csc^2 x$

Useful integrals:

1. $\int \csc x \cot x\,dx = -\csc x + C$	2. $\int \csc^2 x\,dx = -\cot x + C$
3. $\int \cot x\,dx = \ln|\sin x| + C$	4. $\int \csc x\,dx = \ln|\csc x - \cot x| + C.$

Guidelines for Evaluating $\int \cot^m x \csc^n x\,dx$

1. If the power of the $\cot x$, $m = 2k+1$ is odd and positive, save a $\csc x \cot x$ factor and convert the remaining factors to $\csc x$. Then, expand and integrate. We assume that $u = \csc x$, then $du = -\csc x \cot x\,dx$.

$$\int \overbrace{\cot^{2k+1}}^{\text{odd}} x \csc^n x\,dx = \int \overbrace{(\cot^2 x)^k}^{\text{convert to } \csc x} \csc^{n-1} \overbrace{x \csc x \cot x\,dx}^{\text{save for } du}$$

$$= \int (\csc^2 x - 1)^k \csc^{n-1} x \csc x \cot x\,dx = -\int (u^2 - 1)^k u^{n-1}\,du.$$

2. If the power of the $\csc x$, $m = 2k$ is even and positive, save a $\csc^2 x$, factor and convert the remaining factors to $\cot x$. Then, expand and integrate. We assume that $u = \cot x$, then $du = -\csc^2 x\,dx$.

$$\int \cot^m x \overbrace{\csc^{2k}}^{\text{even}} x\,dx = \int \cot^m x \overbrace{(\csc^2 x)^{k-1}}^{\text{convert to } \cot x} \overbrace{\csc^2 x\,dx}^{\text{save for } du}$$

$$= \int \cot^m x\,(1 + \cot^2 x)^{k-1} \csc^2 x\,dx$$

$$= -\int u^m\,(1 + u^2)^{k-1}\,du.$$

3. If the power of the $\cot x$ is an even positive and the power of the $\csc x$ is an odd positive convert to $\csc x$ and use the reduction formula or possibly use integration by parts

Example 9: Find $\int \cot^4 x \csc^4 x\,dx$.

Solution:

$$\begin{aligned}\int \csc^4 x \cot^4 x dx &= \int \csc^2 x \cot^4 x (\csc^2 x)\, dx \\ &= \int (1+\cot^2 x)(\cot^4 x)(\csc^2 x)\, dx \\ &= -\int (\cot^4 x + \cot^6 x)(-\csc^2 x)\, dx \quad u=\cot x \quad du = -\csc^2 x\, dx. \\ &= -\int (u^4+u^6)\, du \\ &= -\frac{u^5}{5} - \frac{u^7}{7} + C \\ &= -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + C.\end{aligned}$$

Integrals Involving Sine-Cosine products with different Angles

Useful identities:

1. $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
2. $\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$
3. $\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

Example 10: Find $\int \sin 4x \cos 3x\, dx$.

Solution:

Use the trigonometric identity $\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$:

$$\begin{aligned}\int \sin 4x \cos 3x\, dx &= \frac{1}{2}\int (\sin x + \sin 7x)\, dx \qquad \int \sin(ax)dx = \frac{-1}{a}\cos(ax) + C \\ &= -\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C.\end{aligned}$$

7.3 Trigonometric Substitution

March 1, 2015

3 Trigonometric Substitution

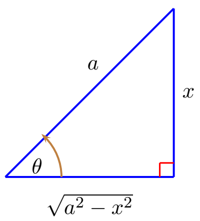
Guidelines for Trigonometric Substitutions

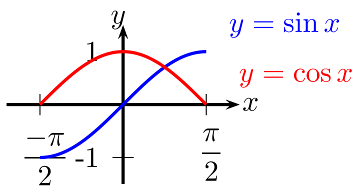
Integrals involving $\sqrt{a^2 - x^2}, (a^2 - x^2)^n$

For Integrals involving $\sqrt{a^2 - x^2}, (a^2 - x^2)^n$, where a is a positive constant. Then use the substitution; $x = a\sin\theta$ where $-\pi/2 \leq \theta \leq \pi/2$, then $dx = a\cos\theta\, d\theta$, and

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta,$$

because, $\cos\theta \geqslant 0$ for $-\pi/2 \leqslant \theta \leqslant \pi/2$.





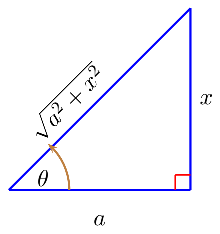
Guidelines for Trigonometric Substitutions

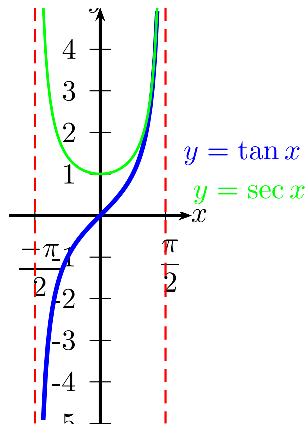
Integrals involving $\sqrt{a^2 + x^2}, (a^2 + x^2)^n$

For Integrals involving $\sqrt{a^2 + x^2}, (a^2 + x^2)^n$, where a is a positive constant. Then use the substitution; $x = a\tan\theta$ where $-\pi/2 < \theta < \pi/2$, then $dx = a\sec^2\theta\, d\theta$, and

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta,$$

because, $\sec\theta \geqslant 0$ for $-\pi/2 < \theta < \pi/2$.





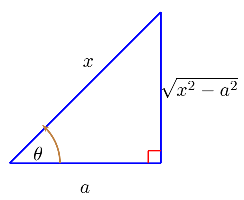
Guidelines for Trigonometric Substitutions

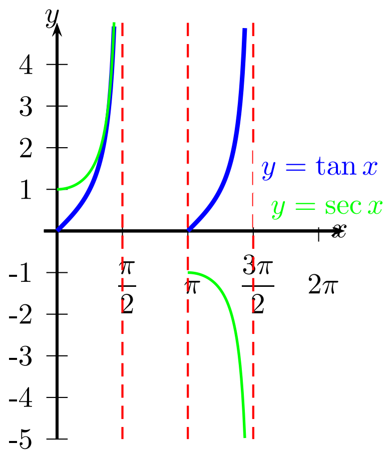
Integrals involving $\sqrt{x^2-a^2}, (x^2-a^2)^n$

For Integrals involving $\sqrt{x^2-a^2}, (x^2-a^2)^n$, where a is a positive constant. Then use the substitution; $x = a\sec\theta$ where $0 \le \theta < \pi/2,\ \pi \le \theta < 3\pi/2$ then $dx = a\sec\theta\tan\theta\, d\theta$, and

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$$

because, $\tan\theta \geqslant 0$ for $0 \leqslant \theta < \dfrac{\pi}{2}$ or $\pi \leqslant \theta < \dfrac{3\pi}{2}$.

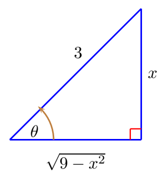




Example 1: Find $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Solution:

$$\text{let } x = 3\sin\theta \text{ then } dx = 3\cos\theta d\theta \qquad \sin\theta = \frac{x}{3}$$
$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\cos\theta. \qquad x^2 = 9\sin^2\theta$$

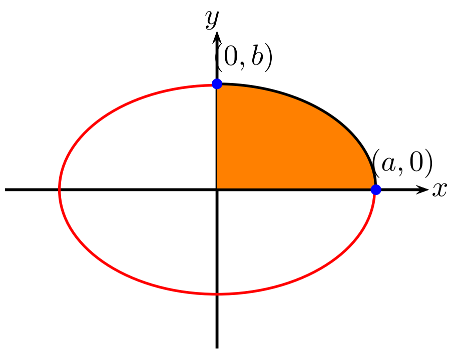


$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta}\cdot 3\cos\theta d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \qquad \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta. \\
&= \int \cot^2\theta\,d\theta \qquad \cot^2\theta = \csc^2\theta - 1. \\
&= \int (\csc^2\theta - 1)\,d\theta \\
&= -\cot\theta - \theta + C \qquad \text{use } \sin\theta = \frac{x}{3} = \frac{\text{opposite}}{\text{hypotenuse}}, \\
&= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.
\end{aligned}$$

Example 2: Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

Solving the equation for y we get $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$ and hence $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ Then $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$. Since the ellipse is symmetric with respect to both axes, then the area A is four times the area in the first quadrant. The part of the ellipse in the first quadrant is given by $y = \frac{b}{a}\sqrt{a^2 - x^2}$, $0 \leq x \leq a$. Hence $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}$, dx.



$$\text{let } x = a\sin\theta \text{ then } dx = a\cos\theta d\theta \;\; \sin\theta = \frac{x}{a}$$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta.$$

$$x = 0, \sin\theta = 0 \Rightarrow \theta = 0x = a \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

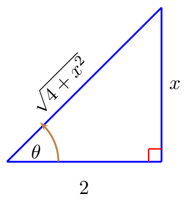
$$\begin{aligned} A &= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx \\ &= 4\frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta a\cos\theta\,d\theta \\ &= 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta && \cos^2\theta = \frac{1+\cos(2\theta)}{2}. \\ &= 4ab\int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\,d\theta \\ &= 2ab\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}} && \text{use } \sin(n\pi) = 0, n \in \mathbb{Z} \\ &= 2ab\left[\frac{\pi}{2} + 0 - 0\right] = ab\pi. \end{aligned}$$

Example 3: Find $\displaystyle\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$.

Solution:

$$\text{Let } x = 2\tan\theta \text{ then } dx = 2\sec^2\theta d\theta \tan\theta = \frac{x}{2}$$

$$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sec\theta. \qquad x^2 = 4\tan^2\theta$$



$$\begin{aligned} \int \frac{1}{x^2\sqrt{4+x^2}}\,dx &= \int \frac{2\sec\theta}{4\tan^2\theta \cdot 2\sec\theta} d\theta \\ &= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta && \frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}. \\ &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta && \text{use } u = \sin\theta\; du = \cos\theta\,d\theta. \\ &= \frac{1}{4}\int \frac{1}{u^2}\,du \\ &= -\frac{1}{4}\frac{1}{u} + C \\ &= -\frac{1}{\sin\theta} + C \\ &= \frac{-\csc\theta}{4} + C \\ &= -\frac{\sqrt{x^2+4}}{4x} + C \end{aligned}$$

Example 4: Find $\displaystyle\int \frac{x}{\sqrt{4+x^2}}\,dx.$

Solution:

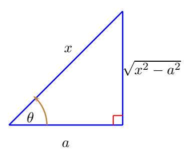
$$\begin{aligned}
\int \frac{x}{\sqrt{4+x^2}}\,dx &= \int (4+x^2)^{-1/2}\,xdx \\
&= \frac{1}{2}\int (4+x^2)^{-1/2}\,2xdx \quad \text{use } \int [f(x)]^n\, f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}+C. \\
&= \frac{1}{2}2(4+x^2)^{1/2}+C \\
&= \sqrt{4+x^2}+C
\end{aligned}$$

Example 5: Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx$, where $a>0$.

Solution:

Let $x = a\sec\theta$ then

$$\begin{aligned}
dx &= a\sec\theta\tan\theta d\theta \qquad\qquad \sec\theta = \frac{x}{a} \\
\sqrt{x^2-a^2} &= \sqrt{a^2\sec^2\theta-a^2} \\
&= \sqrt{a^2(\sec^2\theta-1)} = a\tan\theta. \\
\int \frac{1}{\sqrt{x^2-a^2}}\,dx &= \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta|+C \qquad \text{use } \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}. \\
&= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1 \\
&= \ln|x+\sqrt{x^2-a^2}|-\ln a+C_1 \quad \text{let } C=-\ln a+C_1. \\
&= \ln|x+\sqrt{x^2-a^2}|+C
\end{aligned}$$



Example 6: Evaluate $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$

Solution:

Note that $(4x^2+9)^{3/2} = (\sqrt{4x^2+9})^3 = (\sqrt{(2x)^2+3^2})^3$.

$$\text{Let } 2x = 3\tan\theta \text{ then } x = \frac{3}{2}\tan\theta$$
$$dx = \frac{3}{2}\sec^2\theta d\theta \qquad \tan\theta = \frac{2x}{3}$$
$$\sqrt{4x^2+9} = \sqrt{9\tan^2\theta+9}$$
$$= \sqrt{9(\tan^2\theta+1)} = 3\sec\theta.$$
$$x = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$$
$$x = 3\sqrt{3}/2 \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\frac{3}{2}\sec^2\theta d\theta \qquad \text{use } \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\frac{3}{2}\sec^2\theta = \frac{3}{16}\frac{\tan^3\theta}{\sec\theta}.$$
$$= \int_0^{3\sqrt{3}/2} \frac{3}{16}\frac{\tan^3\theta}{\sec\theta}\,d\theta \qquad \frac{\tan^3\theta}{\sec\theta} = \frac{\sin^3\theta}{\cos^2\theta}.$$
$$= \frac{3}{16}\int_0^{3\sqrt{3}/2} \frac{\sin^2\theta}{\cos^2\theta}\sin\theta\,d\theta \qquad \text{let } u = \cos\theta\ du = -\sin\theta\,d\theta.$$
$$= \frac{3}{16}\int_0^{3\sqrt{3}/2} \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta d\theta \quad \text{if } \theta = 0\ u = 1.$$
$$\text{if } \theta = \pi/3\ u = \frac{1}{2}$$
$$= \frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}(-du)$$
$$= \frac{-3}{16}\int_1^{1/2} [1 - \frac{1}{u^2}]\,du \qquad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$
$$= \frac{3}{16}\int_{1/2}^{1} [1 - \frac{1}{u^2}]\,du$$
$$= \frac{3}{16}\left[u + \frac{1}{u}\right]_{1/2}^{1} = \frac{3}{32}$$

Integrals involve $ax^2 + bx + c$

Complete the square

Example 7: Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.

Solution:

We complete the square $3-2x-x^2 = 3-[x^2+2x+1-1] = 4-(x+1)^2$.

$$\begin{aligned}
\text{Let } x+1 &= 2\sin\theta \text{ then } x = 2\sin\theta - 1 \\
\sin\theta &= \frac{x+1}{2} \quad dx = 2\cos\theta \\
\sqrt{3-2x-x^2} &= \sqrt{4-(x+1)^2} \\
&= \sqrt{4-4\sin^2\theta} = 2\cos\theta.
\end{aligned}$$

$$\begin{aligned}
\int \frac{x}{\sqrt{3-2x-x^2}}\,dx &= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx \\
&= \int \frac{2\sin\theta - 1}{2\cos\theta} 2\cos\theta\, d\theta \\
&= \int [2\sin\theta - 1]\, d\theta \\
&= -2\cos\theta - \theta + C \qquad \text{use } \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}. \\
&= -2\frac{\sqrt{4-(x+1)^2}}{2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C \\
&= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C.
\end{aligned}$$

Algebraic Functions

1. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

2. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

4. $\displaystyle\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C.$

5. $\displaystyle\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|\frac{x+\sqrt{x^2\pm a^2}}{a}\right| + C.$

6. $\displaystyle\int \frac{1}{x\sqrt{a^2\pm x^2}}\,dx = \frac{-1}{a}\ln\left|\frac{a+\sqrt{a^2\pm x^2}}{x}\right| + C.$

7. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

1′. $\displaystyle\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + C$

2′. $\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + C$

3′. $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{f(x)}{a}\right) + C$

4′. $\displaystyle\int \frac{f'(x)}{\sqrt{[f(x)]^2\pm a^2}}\,dx = \ln\left|f(x)+\sqrt{[f(x)]^2\pm a^2}\right| + C.$

5′. $\displaystyle\int \frac{f'(x)}{\sqrt{[f(x)]^2\pm a^2}}\,dx = \ln\left|\frac{f(x)+\sqrt{[f(x)]^2\pm a^2}}{a}\right| + C.$

6′. $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2\pm[f(x)]^2}}\,dx = \frac{-1}{a}\ln\left|\frac{a+\sqrt{a^2\pm[f(x)]^2}}{f(x)}\right| + C.$

7′. $\displaystyle\int \frac{f'(x)}{a^2-[f(x)]^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+f(x)}{a-f(x)}\right| + C.$

7.4 Integration of Rational Functions by Partial Fractions

March 2, 2015

4 Integration Of Rational Functions By Partial Fractions

Let $f(x) = \dfrac{P(x)}{Q(x)}$ be a rational function; that is, $P(x)$ and $Q(x)$ are polynomial functions. If the degree of $P(x)$ is less than the degree of $Q(x)$, we call f a proper rational function. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, we call f an improper rational function. If f is an improper, then by long division, $f(x) = \dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$ where $R(x)/Q(x)$ is a proper rational fraction; that is, the degree of $R(x)$ is less than the degree of $Q(x)$. We know that every proper rational function can be expressed as a sum: $\dfrac{R(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x)$ where $F_1(x), F_2(x), ..., F_n(x)$ are rational functions of the form $\dfrac{Ax+B}{(ax^2+bx+c)^k}$ or $\dfrac{A}{(ax+b)^k}$ in which the denominators are factors of $Q(x)$. The sum is called the partial fraction decomposition of $R(x)/Q(x)$. The first step is finding the form of the partial fraction decomposition of $R(x)/Q(x)$ is to factor $Q(x)$ completely into linear and irreducible quadratic factors, and then collect all repeated factors so that $Q(x)$ is expressed as a product of distinct factors of the form $(ax+b)^m$ and $(ax^2+bx+c)^m$. From these factors we can determine the form of the partial fraction decomposition using the following two rules:

Linear Factor Rule

For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, ..., A_m$ are constants to be determined.

Quadratic Factor Rule

For each factor of the form $(ax^2+bx+c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

where $A_1, A_2, ..., A_m,\ B_1, B_2, ..., B_m,$ are constants to be determined.

Integrating Improper Rational Functions

Example 1: Find $\displaystyle\int \frac{x^3+1}{x-1}\,dx$.

Solution:

By long division $\dfrac{x^3+1}{x-1} = x^2+x+1+\dfrac{2}{x-1}$.

$$\begin{aligned}\int \frac{x^3+1}{x-1}\,dx &= \int (x^2+x+1)\,dx + 2\int \frac{1}{x-1}\,dx \\ &= \frac{1}{3}x^3+\frac{1}{2}x^2+x+2\ln|x-1|+C.\end{aligned}$$

Example 2: Find $\displaystyle\int \frac{x^3+x}{x^2-1}\,dx$

Solution:

By long division we have

$$\begin{aligned}\frac{x^3+x}{x^2-1} &= x+\frac{2x}{x^2-1} \\ \int \frac{x^3+x}{x^2-1}\,dx &= \int \left(x+\frac{2x}{x^2-1}\right)\,dx \\ &= \frac{1}{2}x^2+\ln\left|x^2-1\right|+C.\end{aligned}$$

Integrating Proper Rational Functions

Example 3: Find $\displaystyle\int \frac{3x-17}{x^2-2x-3}\,dx$.

Solution:

Since $x^2-2x-3=(x-3)(x+1)$, then by Linear Factor Rule

$$\begin{aligned}\frac{3x-17}{x^2-2x-3} &= \frac{3x-17}{(x-3)(x+1)} \\ \frac{3x-17}{(x-3)(x+1)} &= \frac{A}{x-3}+\frac{B}{x+1} \qquad \text{Multiply both sides by } (x-3)(x+1). \\ 3x-17 &= A(x+1)+B(x-3).\end{aligned}$$

If $x = 3$, then $A = -2$ and if $x = -1$, then $B = 5$. Thus

$$\begin{aligned}\int \frac{3x-17}{x^2-2x-3}\,dx &= \int \frac{3x-17}{(x-3)(x+1)}\,dx \\ &= \int \left[\frac{-2}{x-3} + \frac{5}{x+1}\right] dx \\ &= -2\int \frac{1}{x-3}\,dx + 5\int \frac{1}{x+1}\,dx \\ &= -2\ln|x-3| + 5\ln|x+1| + C.\end{aligned}$$

Example 4: Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

Solution:

Since $2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$, then by Linear Factor Rule

$$\begin{aligned}\frac{x^2+2x-1}{2x^3+3x^2-2x} &= \frac{x^2+2x-1}{x(2x-1)(x+2)} \\ \frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \qquad \text{Multiply both sides by } x(2x-1)(x+2). \\ x^2+2x-1 &= A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1).\end{aligned}$$

If $x = 0$, then $-1 = -2A$, hence $A = \frac{1}{2}$.

If $x = -2$, then $-1 = 10C$, hence $C = \frac{-1}{10}$.

If $x = \frac{1}{2}$, then $\frac{1}{4} = \frac{5}{4}B$, hence $B = \frac{1}{5}$.

$$\begin{aligned}\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx &= \int \left[\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2}\frac{2}{2x-1} + \frac{-1}{10}\frac{1}{x+2}\right] dx \\ &= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C.\end{aligned}$$

Example 5: Find $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$.

Solution:

Since $x^3+2x^2+x = x(x^2+2x+1) = x(x+1)^2$, then by Linear Factor Rule

$$\begin{aligned}\frac{5x^2+20x+6}{x^3+2x^2+x} &= \frac{5x^2+20x+6}{x(x+1)^2} \\ \frac{5x^2+20x+6}{x(x+1)^2} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \qquad \text{Multiply both sides by } x(x+1)^2. \\ 5x^2+20x+6 &= A(x+1)^2 + Bx(x+1) + Cx.\end{aligned}$$

If $x = 0$, then $6 = A$, hence $A = 6$.

If $x = -1$, then $-9 = -C$, hence $C = 9$.

If $x = 1$, then $31 = 4A + 2B + C$, hence $B = \dfrac{31 - 4(6) - (9)}{2} = -1$.

$$\begin{aligned}\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx &= \int \left[\frac{6}{x} + \frac{-1}{x+1} + \frac{9}{(x+1)^2}\right] dx \\ &= 6\ln|x| - \ln|x+1| + 9\frac{(x+1)^{-1}}{-1} + C \\ &= \ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C.\end{aligned}$$

Example 6: Find $\displaystyle\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2}\,dx$.

Solution:

Since $x^3 - 2x^2 = x^2(x-2)$, then by Linear Factor Rule

$$\begin{aligned}\frac{5x^2 - 3x + 2}{x^3 - 2x^2} &= \frac{5x^2 - 3x + 2}{x^2(x-2)} \\ \frac{5x^2 - 3x + 2}{x^2(x-2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \qquad \text{Multiply both sides by } x^2(x-2). \\ 5x^2 - 3x + 2 &= Ax(x-2) + B(x-2) + Cx^2.\end{aligned}$$

If $x = 0$, then $2 = -2B$, hence $B = -1$.

If $x = 2$, then $16 = 4C$, hence $C = 4$.

If $x = 1$, then $4 = -A - B + C$, hence $A = -4 - (-1) + 4 = 1$.

$$\begin{aligned}\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2}\,dx &= \int \left[\frac{1}{x} + \frac{-1}{x^2} + \frac{4}{x-2}\right] dx \\ &= \int \left[\frac{1}{x} - x^{-2} + \frac{4}{x-2}\right] dx \\ &= \ln|x| + \frac{1}{x} + 4\ln|x-2| + C \\ &= \ln\left|x(x-2)^4\right| + \frac{1}{x} + C.\end{aligned}$$

Example 7: Find $\displaystyle\int \frac{1}{x^2 - a^2}\,dx$, where $a \neq 0$.

Solution:

Since $x^2 - a^2 = (x-a)(x+a)$, then by Linear Factor Rule

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)}$$
$$\frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$ Multiply both sides by $(x-a)(x+a)$.
$$1 = A(x+a) + B(x-a).$$

If $x = a$, then $1 = 2aA$, hence $A = \frac{1}{2a}$.
If $x = -a$, then $1 = -2aB$, hence $B = \frac{-1}{2a}$.

$$\begin{aligned}\int \frac{1}{x^2-a^2}\,dx &= \int \left[\frac{\frac{1}{2a}}{x-a} + \frac{-\frac{1}{2a}}{x+a}\right] dx \\ &= \frac{1}{2a}\int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx \\ &= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C \\ &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.\end{aligned}$$

Example 8: Find $\int \frac{5x^2+11}{x^4+5x^2+4}\,dx$. ***Solution:***
Since $x^4 + 5x^2 + 4 = (x^2+1)(x^2+4)$, then by Quadratic Factor Rule

$$\frac{5x^2+11}{x^4+5x^2+4} = \frac{5x^2+11}{(x^2+1)(x^2+4)}$$
$$\frac{5x^2+11}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$ Multiply both sides by $(x^2+1)(x^2+4)$.
$$5x^2+11 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$
$$5x^2+11 = Ax^3 + Bx^2 + 4Ax + 4B + Cx^3 + Dx^2 + Cx + D$$
$$5x^2+11 = (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D).$$

x^3 coff.	$0 = A + C \Rightarrow A = -C$	(1)
x^2 coff.	$5 = B + D \Rightarrow D = 5 - B$	(2)
x coff.	$0 = 4A + C$	(3)
constant coff.	$11 = 4B + D$	(4)

From (1) and (3) we get $C = 0$ and $A = 0$. From (2) and (4) we get $11 = 4B + 5 - B \Rightarrow B = 2$ & $D = 3$. Thus,

$$\begin{aligned}\int \frac{5x^2+11}{(x^2+1)(x^2+4)}\,dx &= 2\int \frac{1}{x^2+1}\,dx + 3\int \frac{1}{x^2+4}\,dx \\ &= 2\tan^{-1} x + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.\end{aligned}$$

Example 9: Find $\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx$.

Solution:

For the factor x^2-2x+3, we have $a=1, b=-2$, and $c=3$. Hence $D=b^2-4ac=4-4(1)(3)<0$. Then x^2-2x+3, is irreducible. Then by Linear Factor Rule and Quadratic Factor Rule

$$\frac{7x^2-13x+13}{(x-2)(x^2-2x+3)} = \frac{A}{x-2}+\frac{Bx+C}{x^2-2x+3} \quad \text{Multiply both sides by } (x-2)(x^2-2x+3).$$
$$7x^2-13x+13 = A(x^2-2x+3)+(Bx+C)(x-2)$$
$$7x^2-13x+13 = Ax^2-2Ax+3A+Bx^2-2Bx+Cx-2C$$
$$7x^2-13x+13 = (A+B)x^2+(-2A-2B+C)x+(3A-2C).$$

x^2 coff.	$7=A+B$	(1)
x coff.	$-13=-2A-2B+C$	(2)
constant coff.	$13=3A-2C$	(3)
for $x=2$	$15=3A$	(4)

From (4) $A=5$. Hence from (1) $B=2$ and from (3) $C=1$ Thus,

$$\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx$$
$$=\int\left[\frac{5}{x}+\frac{2x+1}{x^2-2x+3}\right]dx \quad \text{Note that } (x^2-2x+3)'=2x-2.$$
$$=\int\left[\frac{5}{x}+\frac{2x+1-2+2}{x^2-2x+3}\right]dx \quad \text{Note that } x^2-2x+3=(x-1)^2+2.$$
$$=\int\left[\frac{5}{x}+\frac{2x-2}{x^2-2x+3}+\frac{3}{(x-1)^2+2}\right]dx \quad \text{Note that } \int\frac{1}{u^2+a^2}\,du=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C.$$
$$=5\ln|x|+\ln(x^2-2x+3)+\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C.$$

■

Example 10: Find $\int \frac{1}{x(x^2+1)^2}\,dx$.

Solution:

We have x a linear factor and $(x^2+1)^2$ an irreducible quadratic factor. Then by Linear Factor Rule and Quadratic Factor Rule

$$\frac{1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2} \quad \text{Multiply both sides by } x(x^2+1)^2.$$
$$1=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x$$
$$1=A(x^4+2x^2+1)+(Bx+C)(x^3+x)+(Dx^2+Ex)$$
$$1=Ax^4+2Ax^2+A+Bx^4+Cx^3+Bx^2+Cx+Dx^2+Ex$$
$$1=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A$$

x^4 coff.	$0 = A + B$	(1)
x^3 coff.	$0 = C$	(2)
x^2 coff.	$0 = 2A + B + D$	(3)
x coff.	$0 = C + E$	(4)
constant coff.	$1 = A$	(5)

From (5) $A = 1$. Hence from (1) $B = -A = -1$. From (2) $C = 0$, and hence from (4) $E = -C = 0$. Finally from (3) $D = -2A - B = -2 + 1 = -1$. Thus,

$$\begin{aligned}
&\int \frac{1}{x(x^2+1)^2}\,dx \\
&= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \qquad \text{Note that } (x^2+1)' = 2x. \\
&= \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int (x^2+1)^{-2}\,2x\,dx \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\frac{(x^2+1)^{-1}}{-1} + C \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C.
\end{aligned}$$

Example 11: Find $\int \frac{x^2+2x+3}{x^2+x+1}\,dx$.

Solution:

Since $\frac{x^2+2x+3}{x^2+x+1}$ is improper we first divide to obtain

$$\begin{aligned}
\frac{x^2+2x+3}{x^2+x+1} &= \frac{(x^2+x+1)+(x+2)}{x^2+x+1} \\
&= \frac{x^2+x+1}{x^2+x+1} + \frac{x+2}{x^2+x+1} \\
&= 1 + \frac{x+2}{x^2+x+1}.
\end{aligned}$$

Note that the quadratic $x^2 + x + 1$ is irreducible because its discriminant $D = b^2 - 4ac = 1 - 4 = -3 < 0$. It can not be factor. So we complete the square

$$\begin{aligned}
x^2+x+1 &= x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 \\
&= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \\
&= \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2
\end{aligned}$$

$$\begin{aligned}
x+2 &= x+\frac{1}{2}+\frac{3}{2} \\
\frac{x+2}{x^2+x+1} &= \frac{x+\frac{1}{2}+\frac{3}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \\
&= \frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{\frac{3}{2}}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}
\end{aligned}$$

$$\begin{aligned}
&\int \frac{x^2+2x+3}{x^2+x+1}\,dx \\
&= \int \left[1+\frac{x+2}{x^2+x+1}\right] dx \\
&= \int \left[1+\frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{\frac{3}{2}}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right] dx \\
&= \int dx+\int \frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx+\int \frac{\frac{3}{2}}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx \\
&= \int dx+\frac{1}{2}\int \frac{2\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx+\frac{3}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx \\
&= x+\frac{1}{2}\ln\left(\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right)+\frac{3}{2}\,\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C \\
&= x+\frac{1}{2}\ln\left(x^2+x+1\right)+\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C.
\end{aligned}$$

Note that $((x+1/2)^2)' = 2(x+1/2)$.

Rationalizing Substitutions

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u=\sqrt[n]{g(x)}$ my be effective. Also if the integrand contains an expression of the form $\sqrt[n]{f(x)}$, and $\sqrt[m]{f(x)}$, then the substitution $u=\sqrt[mn]{f(x)}$ my be effective.

Example 12: Find $\displaystyle\int \frac{1}{1+\sqrt{2x}}\,dx$.

Solution:

$$\text{Let } u = \sqrt{2x} \qquad u^2 = 2x$$
$$2u\,du = 2\,dx \qquad u\,du = dx.$$

$$\begin{aligned}
\int \frac{1}{1+\sqrt{2x}}\,dx &= \int \frac{1}{1+u}u\,du \qquad \text{Note that } u = u+1-1.\\
&= \int \frac{u+1-1}{1+u}\,du\\
&= \int \left[1 - \frac{1}{u+1}\right]dx\\
&= u - \ln|u+1| + C\\
&= \sqrt{2x} - \ln(1+\sqrt{2x}) + C.
\end{aligned}$$

Example 13: Find $\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}\,dx$.

Solution:

$$\text{Let } u = \sqrt{x}\sqrt[3]{x} = \sqrt[6]{x} \qquad u^6 = x$$
$$6u^5\,du = dx \qquad \sqrt{x} = u^3 \quad \sqrt[3]{x} = u^2.$$

$$\begin{aligned}
\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}\,dx &= \int \frac{1}{u^3(1+u^2)}6u^5\,du\\
&= 6\int \frac{u^5}{u^3(1+u^2)}\,du\\
&= 6\int \frac{u^2}{1+u^2}\,du \qquad u^2 = u^2+1-1.
\end{aligned}$$

$$6\int \frac{u^2+1-1}{1+u^2}\,du$$

$$\begin{aligned}
&= 6\int \left[1 - \frac{1}{u^2+1}\right]dx\\
&= 6u - 6\tan^{-1}u + C\\
&= 6\sqrt[6]{x} - 6\tan^{-1}(\sqrt[6]{x}) + C.
\end{aligned}$$